Table 12.3

Number of independent variables	Explicit form	Implicit form	Graph resides in . . .
1	$y = f(x)$	$F(x, y) = 0$	$\mathbf{R}^2$ (xy-plane)
2	$z = f(x, y)$	$F(x, y, z) = 0$	$\mathbf{R}^3$ (xyz-space)
3	$w = f(x, y, z)$	$F(w, x, y, z) = 0$	$\mathbf{R}^4$
n	$y = f(x_1, x_2, \ldots, x_n)$	$F(x_1, x_2, \ldots, x_n, x_{n+1}) = 0$	$\mathbf{R}^{n+1}$

The concepts of domain and range extend from the one- and two-variable cases in an obvious way.

DEFINITION Function, Domain, and Range with n Independent Variables

The **function** $y = f(x_1, x_2, \ldots, x_n)$ assigns a unique real number y to each point $(x_1, x_2, \ldots, x_n)$ in a set D in $\mathbf{R}^n$. The set D is the **domain** of f. The **range** is the set of real numbers y that are assumed as the points $(x_1, x_2, \ldots, x_n)$ vary over the domain.

EXAMPLE 7 Finding domains Find the domain of the following functions.

a. $g(x, y, z) = \sqrt{16 - x^2 - y^2 - z^2}$ **b.** $h(x, y, z) = \dfrac{12y^2}{z - y}$

SOLUTION

a. Values of the variables that make the argument of a square root negative must be excluded from the domain. In this case, the quantity under the square root is nonnegative provided

$$16 - x^2 - y^2 - z^2 \geq 0, \quad \text{or} \quad x^2 + y^2 + z^2 \leq 16.$$

Therefore, the domain of g is a closed ball in $\mathbf{R}^3$ of radius 4.

➤ Recall that a closed ball of radius r is the set of all points on or within a sphere of radius r.

b. Values of the variables that make a denominator zero must be excluded from the domain. In this case, the denominator vanishes for all points in $\mathbf{R}^3$ that satisfy $z - y = 0$, or $y = z$. Therefore, the domain of h is the set $\{(x, y, z): y \neq z\}$. This set is $\mathbf{R}^3$ excluding the points on the plane $y = z$.

Related Exercises 42–48 ◀

QUICK CHECK 9 What is the domain of the function $w = f(x, y, z) = 1/xyz$? ◀

Graphs of Functions of More Than Two Variables

Graphing functions of *two* independent variables requires a three-dimensional coordinate system, which is the limit of ordinary graphing methods. Clearly, difficulties arise in graphing functions with three or more independent variables. For example, the graph of the function $w = f(x, y, z)$ resides in four dimensions. Here are two approaches to representing functions of three independent variables.

The idea of level curves can be extended. With the function $w = f(x, y, z)$, level curves become **level surfaces**, which are surfaces in $\mathbf{R}^3$ on which w is constant. For example, the level surfaces of the function

$$w = f(x, y, z) = \sqrt{z - x^2 - 2y^2}$$

satisfy $w = \sqrt{z - x^2 - 2y^2} = C$, where C is a nonnegative constant. This equation is satisfied when $z = x^2 + 2y^2 + C^2$. Therefore, the level surfaces are elliptic paraboloids stacked one inside another (Figure 12.35).

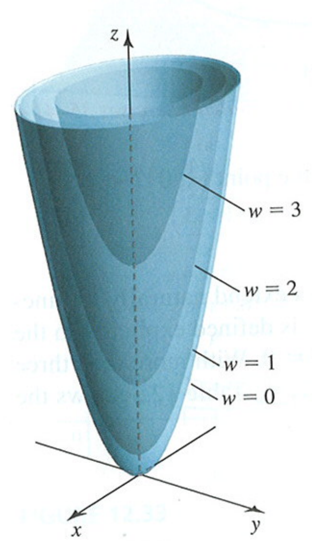


FIGURE 12.35

Another approach to displaying functions of three variables is to use colors to gain access to the fourth dimension. Figure 12.36a shows the electrical activity of the heart at one snapshot in time. The three independent variables correspond to locations in the heart. At each point, the value of the electrical activity, which is the dependent variable, is coded by colors.

In Figure 12.36b, the dependent variable is the switching speed in an integrated circuit, again represented by colors, as it varies over points of the domain. Software to produce such images, once expensive and inefficient, has become much more accessible.

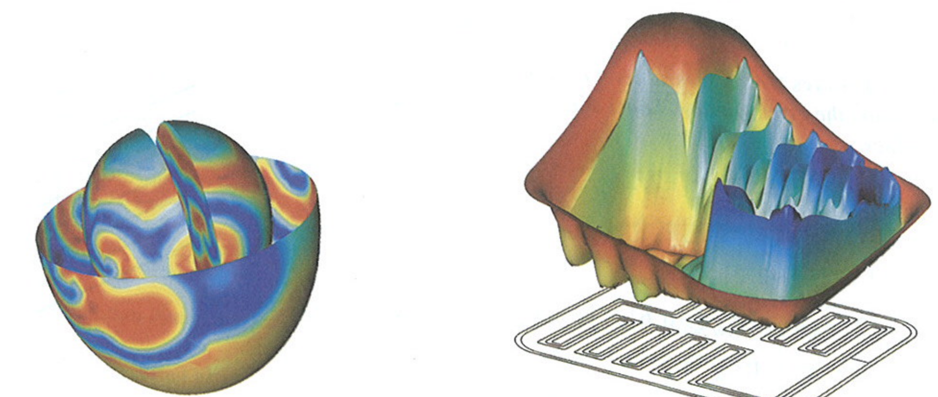


FIGURE 12.36 (a) (b)

SECTION 12.2 EXERCISES

Review Questions

1. A function is defined by $z = x^2y - xy^2$. Identify the independent and dependent variables.
2. What is the domain of $f(x, y) = x^2y - xy^2$?
3. What is the domain of $g(x, y) = 1/(xy)$?
4. What is the domain of $h(x, y) = \sqrt{x - y}$?
5. How many axes or how many dimensions are needed to graph the function $z = f(x, y)$? Explain.
6. Explain how to graph the level curves of a surface $z = f(x, y)$.
7. Describe in words the level curves of the paraboloid $z = x^2 + y^2$.
8. How many axes (or how many dimensions) are needed to graph the level surfaces of $w = f(x, y, z)$? Explain.
9. The domain of $Q = f(u, v, w, x, y, z)$ lies in $\mathbf{R}^n$ for what value of n? Explain.
10. Give two methods for graphically representing a function with three independent variables.

Basic Skills

11–18. Domains *Find the domain of the following functions.*

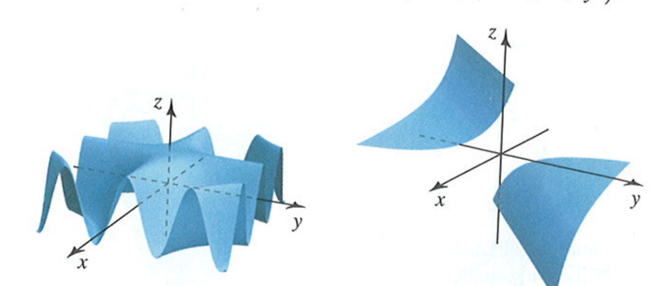
11. $f(x, y) = 2xy - 3x + 4y$
12. $f(x, y) = \cos(x^2 - y^2)$
13. $f(x, y) = \sin\left(\dfrac{x}{y}\right)$
14. $f(x, y) = \dfrac{12}{y^2 - x^2}$
15. $g(x, y) = \ln(x^2 - y)$
16. $f(x, y) = \tan^{-1}(x + y)$
17. $g(x, y) = \sqrt{\dfrac{xy}{x^2 + y^2}}$
18. $h(x, y) = \sqrt{x - 2y + 4}$

19–26. Graphs of familiar functions *Use what you learned about surfaces in Section 12.1 to sketch a graph of the following functions. In each case identify the surface, and state the domain and range of the function.*

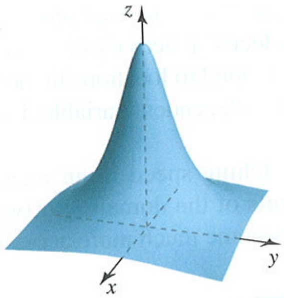
19. $f(x, y) = 3x - 6y + 18$
20. $h(x, y) = 2x^2 + 3y^2$
21. $p(x, y) = x^2 - y^2$
22. $F(x, y) = \sqrt{1 - x^2 - y^2}$
23. $G(x, y) = -\sqrt{1 + x^2 + y^2}$
24. $H(x, y) = \sqrt{x^2 + y^2}$
25. $P(x, y) = \sqrt{x^2 + y^2 - 1}$
26. $g(x, y) = y^3 + 1$

27. **Matching surfaces** Match functions a–d with surfaces A–D in the figure.

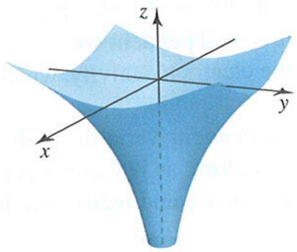
a. $f(x, y) = \cos xy$ **b.** $g(x, y) = \ln(x^2 + y^2)$
c. $h(x, y) = 1/(x - y)$ **d.** $p(x, y) = 1/(1 + x^2 + y^2)$



(A) (B)



(C)



(D)

28–33. Level curves *Graph several level curves of the following functions using the given window. Label at least two level curves with their z-values.*

28. $z = 2x - y$; $[-2, 2] \times [-2, 2]$

29. $z = \sqrt{x^2 + 4y^2}$; $[-8, 8] \times [-8, 8]$

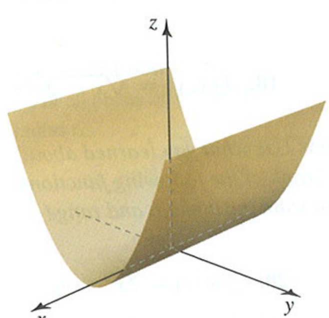
30. $z = e^{-x^2-2y^2}$; $[-2, 2] \times [-2, 2]$

31. $z = \sqrt{25 - x^2 - y^2}$; $[-6, 6] \times [-6, 6]$

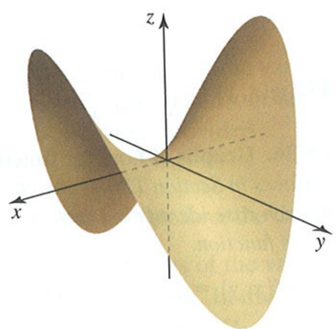
32. $z = \sqrt{y - x^2 - 1}$; $[-5, 5] \times [-5, 5]$

33. $z = 3\cos(2x + y)$; $[-2, 2] \times [-2, 2]$

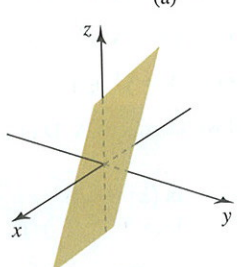
34. Matching level curves with surfaces Match surfaces a–f in the figure with level curves A–F.



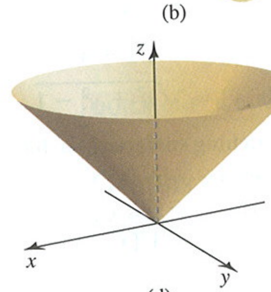
(a)



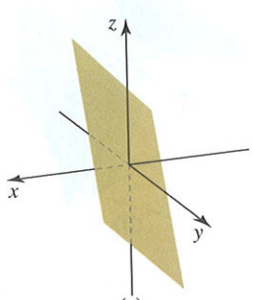
(b)



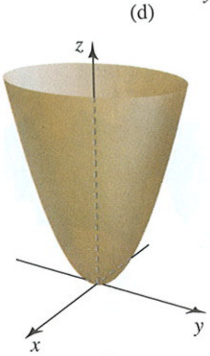
(c)



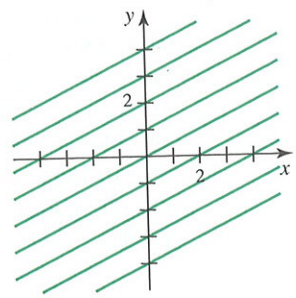
(d)



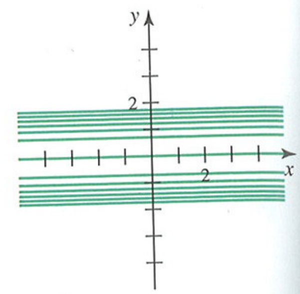
(e)



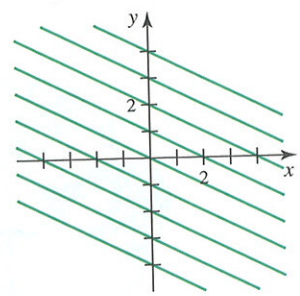
(f)



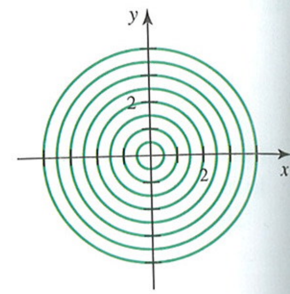
(A)



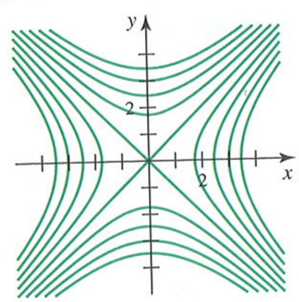
(B)



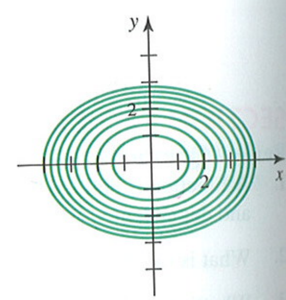
(C)



(D)



(E)



(F)

35. A volume function The volume of a right circular cone of radius r and height h is $V(r, h) = \pi r^2 h/3$.

a. Graph the function in the window $[0, 5] \times [0, 5] \times [0, 150]$.

b. What is the domain of the volume function?

c. What is the relationship between the values of r and h when $V = 100$?

36. Earned run average A baseball pitcher's earned run average (ERA) is $A(e, i) = 9e/i$, where e is the number of earned runs given up by the pitcher and i is the number of innings pitched. Good pitchers have low ERAs. Assume that $e \geq 0$ and $i > 0$ are real numbers.

a. The single-season major league record for the lowest ERA was set by Dutch Leonard of the Detroit Tigers in 1914. During that season, Dutch pitched a total of 224 innings and gave up just 24 earned runs. What was his ERA?

b. Determine the ERA of a relief pitcher who gives up 4 earned runs in one-third of an inning.

c. Graph the level curve $A(e, i) = 3$, and describe the relationship between e and i in this case.

37. Electric potential function The electric potential function for two positive charges, one at $(0, 1)$ with twice the strength as the charge at $(0, -1)$, is given by

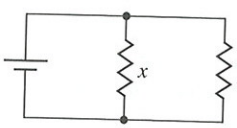
$$\varphi(x, y) = \frac{2}{\sqrt{x^2 + (y - 1)^2}} + \frac{1}{\sqrt{x^2 + (y + 1)^2}}.$$

a. Graph the electric potential using the window $[-5, 5] \times [-5, 5] \times [0, 10]$.
b. For what values of x and y is the potential φ defined?
c. Is the electric potential greater at $(3, 2)$ or $(2, 3)$?
d. Describe how the electric potential varies along the line $y = x$.

38. Cobb-Douglas production function The output Q of an economic system subject to two inputs, such as labor L and capital K, is often modeled by the Cobb-Douglas production function $Q(L, K) = cL^aK^b$, where a, b, and c are positive real numbers. When $a + b = 1$, the case is called *constant returns to scale*. Suppose $a = \frac{1}{3}$, $b = \frac{2}{3}$, and $c = 40$.

a. Graph the output function using the window $[0, 20] \times [0, 20] \times [0, 500]$.
b. If L is held constant at $L = 10$, write the function that gives the dependence of Q on K.
c. If K is held constant at $K = 15$, write the function that gives the dependence of Q on L.

39. Resistors in parallel Two resistors wired in parallel in an electrical circuit give an effective resistance of $R(x, y) = \dfrac{xy}{x + y}$, where x and y are the positive resistances of the individual resistors (typically measured in ohms).



a. Graph the resistance function using the window $[0, 10] \times [0, 10] \times [0, 5]$.
b. Estimate the maximum value of R for $0 < x \leq 10$ and $0 < y \leq 10$.
c. Explain what it means to say that the resistance function is symmetric in x and y.

40. Water waves A snapshot of a water wave moving toward shore is described by the function $z = 10 \sin(2x - 3y)$, where z is the height of the water surface above (or below) the xy-plane, which is the level of undisturbed water.

a. Graph the height function using the window $[-5, 5] \times [-5, 5] \times [-15, 15]$.
b. For what values of x and y is z defined?
c. What are the maximum and minimum values of the water height?
d. Give a vector in the xy-plane that is orthogonal to the level curves of the crests and troughs of the wave (which also gives the direction of wave propagation).

41. Approximate mountains Suppose the elevation of Earth's surface over a 16-mi by 16-mi region is approximated by the function

$$z = 10e^{-(x^2+y^2)} + 5e^{-((x+5)^2+(y-3)^2)/10} + 4e^{-2((x-4)^2+(y+1)^2)}.$$

a. Graph the height function using the window $[-8, 8] \times [-8, 8] \times [0, 15]$.
b. Approximate the points (x, y) where the peaks in the landscape appear.
c. What are the approximate elevations of the peaks?

42–48. Domains of functions of three or more variables *Find the domain of the following functions. If possible, give a description of the domain in words (for example, all points outside a sphere of radius 1 centered at the origin).*

42. $f(x, y, z) = 2xyz - 3xz + 4yz$

43. $g(x, y, z) = \dfrac{1}{x - z}$

44. $p(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 9}$

45. $f(x, y, z) = \sqrt{y - z}$

46. $Q(x, y, z) = \dfrac{10}{1 + x^2 + y^2 + 4z^2}$

47. $F(x, y, z) = \sqrt{y - x^2}$

48. $f(w, x, y, z) = \sqrt{1 - w^2 - x^2 - y^2 - z^2}$

Further Explorations

49. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The domain of the function $f(x, y) = 1 - |x - y|$ is $\{(x, y): x \geq y\}$.
b. The domain of the function $Q = g(w, x, y, z)$ is a region in $\mathbf{R}^3$.
c. All level curves of the plane $z = 2x - 3y$ are lines.

50–56. Graphing functions

a. Determine the domain and range of the following functions.
b. Graph each function using a graphing utility. Be sure to experiment with the window and orientation to give the best perspective on the surface.

50. $g(x, y) = e^{-xy}$

51. $f(x, y) = |xy|$

52. $p(x, y) = 1 - |x - 1| + |y + 1|$

53. $h(x, y) = (x + y)/(x - y)$

54. $G(x, y) = \ln[2 + \sin(x + y)]$

55. $F(x, y) = \tan^2(x - y)$

56. $P(x, y) = \cos x \sin 2y$

57–60. Peaks and valleys *The following functions have exactly one isolated peak or one isolated depression (one local maximum or minimum). Use a graphing utility to approximate the coordinates of the peak or depression.*

57. $f(x, y) = x^2y^2 - 8x^2 - y^2 + 6$

58. $g(x, y) = (x^2 - x - 2)(y^2 + 2y)$

59. $h(x, y) = 1 - e^{-(x^2+y^2-2x)}$

60. $p(x, y) = 2 + |x - 1| + |y - 1|$

61. Level curves of planes Prove that the level curves of the plane $ax + by + cz = d$ are parallel lines in the xy-plane, provided $a^2 + b^2 \neq 0$ and $c \neq 0$.

Applications

62. Level curves of a savings account Suppose you make a one-time deposit of P dollars into a savings account that earns interest at an annual rate of $p\%$ compounded continuously. The balance in the account after t years is $B(P, r, t) = Pe^{rt}$, where $r = p/100$ (for example, if the annual interest rate is 4%, then $r = 0.04$). Let the interest rate be fixed at $r = 0.04$.

a. With a target balance of \$2000, find the set of all points (P, t) that satisfy $B = 2000$. This curve gives all deposits P and times t that result in a balance of \$2000.

b. Repeat part (a) with $B = \$500, \$1000, \$1500$, and $\$2500$, and draw the resulting level curves of the balance function.

c. In general, on one level curve, if t increases, does P increase or decrease?

63. Level curves of a savings plan Suppose you make monthly deposits of P dollars into an account that earns interest at a *monthly* rate of $p\%$. The balance in the account after t years is $B(P, r, t) = P\left[\frac{(1 + r)^{12t} - 1}{r}\right]$, where $r = p/100$ (for example, if the annual interest rate is 9%, then $p = \frac{9}{12} = 0.75$ and $r = 0.0075$). Let the time of investment be fixed at $t = 20$ years.

a. With a target balance of \$20,000, find the set of all points (P, r) that satisfy $B = 20{,}000$. This curve gives all deposits P and monthly interest rates r that result in a balance of \$20,000 after 20 years.

b. Repeat part (a) with $B = \$5000, \$10{,}000, \$15{,}000$, and $\$25{,}000$, and draw the resulting level curves of the balance function.

64. Quarterback ratings One measurement of the quality of a quarterback in the National Football League is known as the *quarterback rating*. The rating formula is $R(c, t, i, y) = \frac{50 + 20c + 80t - 100i + 100y}{24}$, where c is the percentage of passes completed, t is the percentage of passes thrown for touchdowns, i is the percentage of intercepted passes, and y is the yards gained per attempted pass.

a. In his career, Hall of Fame quarterback Johnny Unitas completed 54.57% of his passes, 5.59% of his passes were thrown for touchdowns, 4.88% of his passes were intercepted, and he gained an average of 7.76 yards per attempted pass. What was his quarterback rating?

b. If c, t, and y remained fixed, what happens to the quarterback rating as i increases? Explain your answer with and without mathematics.

[*Source: The College Mathematics Journal* (November 1993).]

65. Ideal Gas Law Many gases can be modeled by the Ideal Gas Law, $PV = nRT$, which relates the temperature (T, measured in Kelvin (K)), pressure (P, measured in Pascals (Pa)), and volume (V, measured in m^3) of a gas. Assume that the quantity of gas in question is $n = 1$ mole (mol). The gas constant has a value of $R = 8.3$ m^{3}Pa/mol-K.

a. Consider T to be the dependent variable and plot several level curves (called *isotherms*) of the temperature surface in the region $0 \le P \le 100{,}000$ and $0 \le V \le 0.5$.

b. Consider P to be the dependent variable and plot several level curves (called *isobars*) of the pressure surface in the region $0 \le T \le 900$ and $0 < V \le 0.5$.

c. Consider V to be the dependent variable and plot several level curves of the volume surface in the region $0 \le T \le 900$ and $0 < P \le 100{,}000$.

Additional Exercises

66–69. Challenge domains *Find the domains of the following functions. Specify the domain mathematically and then describe it in words or with a sketch.*

66. $g(x, y, z) = \dfrac{10}{x^2 - (y + z)x + yz}$

67. $f(x, y) = \sin^{-1}(x - y)^2$

68. $f(x, y, z) = \ln(z - x^2 - y^2 + 2x + 3)$

69. $h(x, y, z) = \sqrt[4]{z^2 - xz + yz - xy}$

70. Other balls The closed unit ball in $\mathbf{R}^3$ centered at the origin is the set $\{(x, y, z): x^2 + y^2 + z^2 \le 1\}$. Describe in words the following alternative unit balls.

a. $\{(x, y, z): |x| + |y| + |z| \le 1\}$

b. $\{(x, y, z): \max\{|x|, |y|, |z|\} \le 1\}$, where $\max\{a, b, c\}$ is the maximum value of a, b, and c.

QUICK CHECK ANSWERS

1. $\mathbf{R}^2$; $\{(x, y): y \ge 0\}$ **2.** No; no **3.** $z = -\sqrt{1 + x^2 + y^2}$ **4.** No, otherwise the function would have two values at a single point. **5.** Concentric circles **6.** No; $z = 0$ is not in the range of the function. **7.** 0.97 **8.** 8 **9.** $\{(x, y, z): x \ne 0 \text{ and } y \ne 0 \text{ and } z \ne 0\}$ (which is $\mathbf{R}^3$, excluding the coordinate planes)

12.3 Limits and Continuity

You have now seen examples of functions of several variables, but calculus has not yet entered the picture. In this section we revisit topics encountered in single-variable calculus and see how they apply to functions of several variables. We begin with the fundamental concepts of limits and continuity.